

External Rays

Specialize: $f \in \mathbb{C}[z]$ noni, deg d ,

Assume $K(f)$ is connected

Böttcher:
$$\begin{array}{ccc} \hat{\mathbb{C}} \setminus \bar{D} & \xrightarrow{\gamma} & \hat{\mathbb{C}} \setminus K \\ \downarrow w & & \downarrow f \\ w^{-1} \hat{\mathbb{C}} \setminus \bar{D} & \xrightarrow{\gamma} & \hat{\mathbb{C}} \setminus K \end{array}$$

External rays

$$R_\theta = \{ \gamma(e^{2\pi i \theta}) \mid r > 1 \}$$

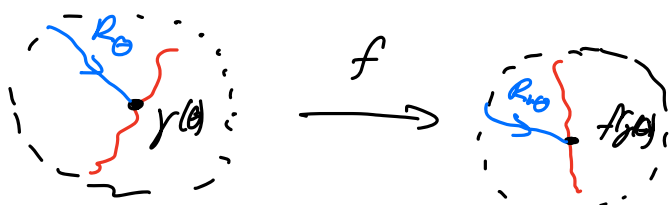
- For ac. θ , R_θ lands at a point $\gamma(\theta) \in J$
- γ extends continuously to $\hat{\mathbb{C}} \setminus D \rightarrow \overline{\hat{\mathbb{C}} \setminus K}$ iff J is locally connected
- $\gamma: S^1 \rightarrow J$ is a homeo iff J is a Jordan curve

$$* E: \underset{\mathbb{R}/\mathbb{Z}}{S^1} \rightarrow \underset{\mathbb{R}/\mathbb{Z}}{S^1}, E(\theta) = u\theta$$

Prop. Suppose R_θ lands at $\gamma(\theta) \in J$. Then

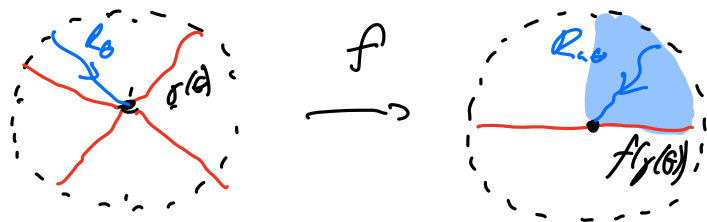
- 1) $R_{u\theta}$ lands at $\gamma(u\theta) = f(\gamma(\theta))$.
- 2) Each ray $R_{(\frac{\theta+i}{n})}$ lands at a point of $f^{-1}(\gamma(\theta))$.
- 3) Every point of $f^{-1}(\gamma(\theta))$ is the landing point of some ray $R_{(\frac{\theta+i}{n})}$.

Proof. $f'(y(\theta)) \neq 0$: Local inv First, limit argument



So by cont R_{θ} land at $f(y(\theta))$, and any ray at $f(y(\theta))$ pull back to one at $y(\theta)$.

If $f'(y(\theta)) = 0$, branched: R_{θ} land at $f(y(\theta))$, and any ray at $f(y(\theta))$ pull back to several at $y(\theta)$.



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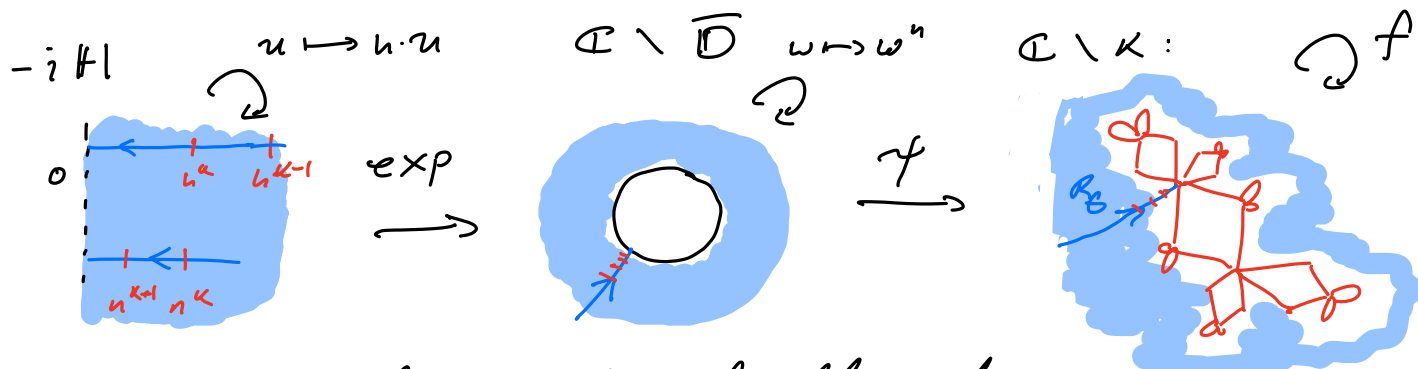
Thm. (Rational Rays)

- 1) If $\theta \in \mathbb{Q}/\mathbb{Z}$, then R_{θ} land.
- 2.1) If $\theta \in \mathbb{Q}/\mathbb{Z}$ is periodic of period p , then $y(\theta)$ is periodic of period k/p .
- 2.2) The point $y(\theta)$ is either repelling or a primitive $(p/k)^{\text{th}}$ root of unity.
- 2.3) Only finitely many rays land at $y(\theta)$ and they all have the same period.
- 3) If $\theta \in \mathbb{Q}/\mathbb{Z}$ is preperiodic of preperiod q , then the preperiod of $y(\theta)$ is q .

Proof. $R_\theta(t) := \psi(e^{t-2\pi i\theta})$,

$$R_\theta[t_0, t_1] = \{R_\theta(t) \mid t_0 \leq t \leq t_1\}$$

Minimal cover:



f is a local isometry for the hyperbolic metric

Segments $[u^{-(k+1)} + i\theta, u^{-k} + i\theta]$ have hyperbolic length $\log u$ and are mapped into each other by $u \mapsto u \cdot u$.

$$R_\theta = \bigcup_k I_k, \quad I_k = R_\theta[u^{-(k+1)}, u^{-k}]$$

$f: I_k \rightarrow I_{k-1}$ isometrically

Lemma. $\text{diam}_{\text{Euc.}} I_k \rightarrow 0$ unif in θ as $k \rightarrow \infty$

Proof. As $z \rightarrow \partial K$, hyperbolic metric shrinks unif.

1) wlog $E(\theta) = \theta$

if p is an accumulation pt of R_θ , then $f(p) = p$.

Set of accumulation pts is connected + only fin. many fixed pts $\Rightarrow R_\theta$ levels at p .

2.1) By prev. prop.

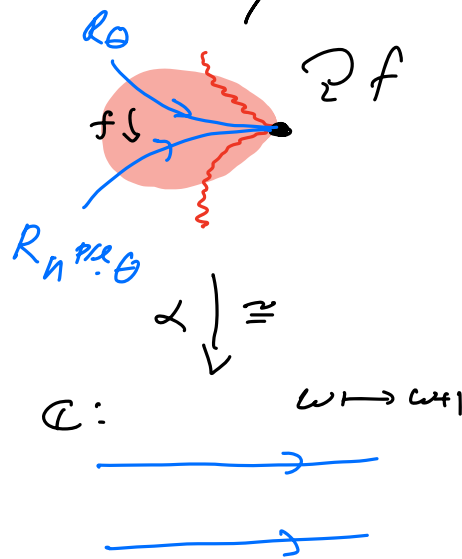
2.2) Back to the proof of 1):

$f^{-1}: I_k \rightarrow I_{k+1}$, so by I. Weil, the multiplier is > 1 or $= 1$.

• If $E^P(\theta) = \theta$, $f^k(\gamma(\theta)) = \gamma(\theta)$, mult λ .

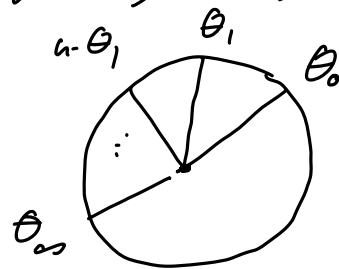
$\lambda^{P/k} = 1$. If not primitive, then $\lambda^{P/k} = 1$ for $k \mid P$.

• $f^{k \cdot \frac{P}{k}} = f^{P/k}$ maps each repelling petal to itself. If $k \neq 1$, $f^{P/k}(R_\theta) \neq R_\theta$, but under the linearization, they must be the same.



2.3) $\theta_0 \in \mathbb{Q}/\mathbb{Z}$, $X = \{\theta \mid R_\theta \text{ lands at } \gamma(\theta_0)\}$

• $E: X \rightarrow X$ is surj, and since $f'(\gamma(\theta_0)) \neq 0$, it is injective and preserves cyclic order of rays.

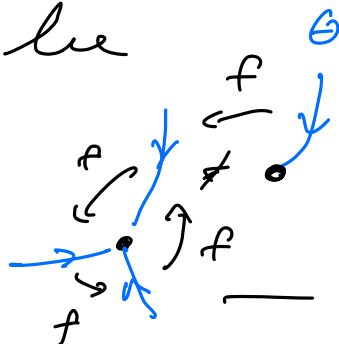


• Let $\theta_1 \in X$. Since f preserves cyclic order in X , $E^k(\theta_1) \rightarrow \theta_\infty$, $E(\theta_\infty) = \theta_\infty$. But on $\mathcal{S}'$,

E is repelling, so the sequ. interval eventually
 const. diverges $\Rightarrow E(\theta, \cdot) = \theta$, non-periodic.

3) $E^{k+1}(\theta) = E(\theta)$, but $E(\theta) \neq \theta$.

If $\gamma(\theta) = \gamma(E(\theta)) = f(\gamma(\theta))$, but f not be
 inj. $\Rightarrow$ lands on the cycle.



Thm. $f \in \mathbb{C}[z]$, suppose $\mathcal{J}(f)$ is loc.
 connected. Then:

- 1) Every periodic point is repelling or
 parabolic.
- 2) The boundary of every Siegel disk
 is a Jordan curve. (Open whether there
 are any wild ones)

* If a polynomial has a Cremona point,
 then $\mathcal{J}(f)$ is not locally connected.

Proof of the Thm. $\mathcal{J}$ locally conn.

1) $f(z_0) = z_0$

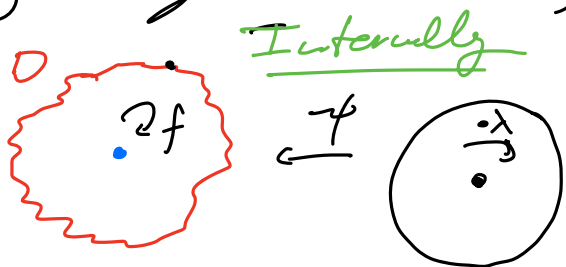
$X = \{ \theta \in \mathbb{R}/\mathbb{Z} \mid R_\theta \text{ lands at } z_0 \} = \gamma^{-1}(z_0)$ is

closed and nonempty. Since $f'(z_0) \neq 0$, f is
 loc. iso and some arg from before shows that

X can't have as many points.

$\Rightarrow X$ is finite $\Rightarrow X \subset \mathbb{Q}/\mathbb{Z}$ periodic $\Rightarrow z_0$ is parabolic or repelling.

2) Angel disk: D , $\lambda = e^{2\pi i t}$, $t \notin \mathbb{Q}$



Internally

Since $\partial D \subset J$ is l.c., $\gamma: S^1 \rightarrow \partial D$.

$$\text{if } \gamma(\theta_1) = \gamma(\theta_2) = \gamma(\theta_1 + \theta_2 - \theta_1)$$

$$\gamma(k + \theta_1) = \gamma(k + \theta_1 - \theta_1)$$

$$\{k + \theta_1\} = \mathbb{R}/\mathbb{Z} \Rightarrow \gamma(\theta) = \gamma(\theta + \theta_2 - \theta_1) \quad \forall \theta.$$

$\hookrightarrow \theta \in \mathbb{Q}$:
in $\mathbb{C}$:

$\therefore \overline{D} \cong \overline{D}/\sim_\gamma$ can't be embedded

$\leftarrow \theta \notin \mathbb{Q}$:

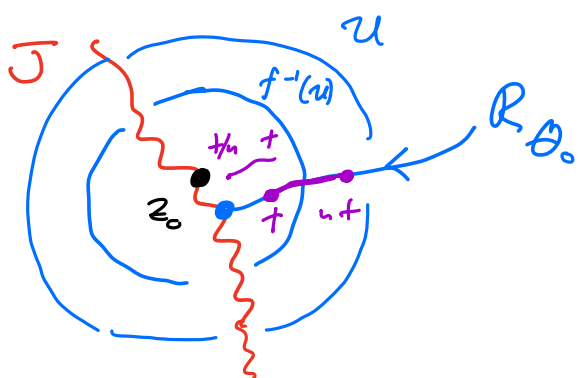
$$\partial D \cong *$$

$\Rightarrow \text{inj} \Rightarrow \text{homeo}$



Thm. Every repelling or parabolic periodic point is the landing point of a rational ray.

Proof: 1) Repelling: $f(z_0) = z_0$, $|\lambda| > 1$



Pick a Koenigs neighborhood

$$U \ni \alpha, \quad f^{-1}: U \rightarrow U.$$

By Lemma, choose $t \ll 0$ s.t.

$$(*) \text{diam}_{Ecc} R_\theta [t, t+\epsilon] < d(f^{-1}(u), \partial U) \neq 0.$$

- R_θ land at $\gamma(0)$, preimages of $\gamma(0)$ are dense in J , so $\exists R_{\theta_0}$ landing in $f^{-1}(u)$
 $R_{\theta_0}(t) \in f^{-1}(u) \Rightarrow R_{\theta_0}[t, t+\epsilon] \subset U$
- Pick $\theta_i \in \mathbb{R}/\mathbb{Z}$ s.t. $E(\theta_i) = \theta_{i-1}$, $R_{\theta_i} \in f^{-1}(u)$.

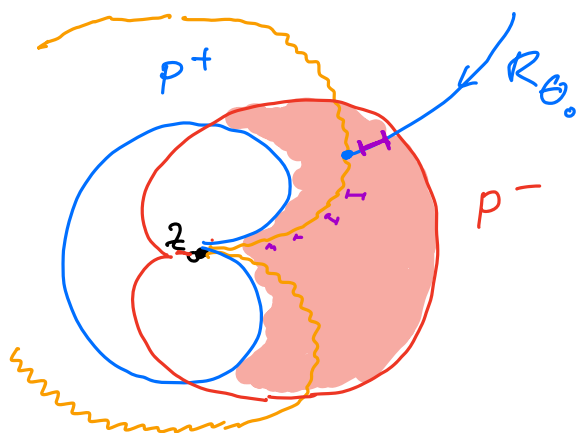
$\leadsto R_{\theta_i}[u^{-i}t, u^{-(i+1)}t] \subset f^{-i}(u)$, extending:

$$(*) \leadsto R_{\theta_i}[u^{-i}t, u^{-(j-1)}t] \subset f^{-j}(u) \rightarrow z_0$$

- Take a limit point $\theta_i \rightarrow \theta$:

$R_\theta(-\infty, u^{-(j-1)}t] \subset f^{-j}(u)$, so R_θ land at z_0 .

2) Parabolic: Use the region in a repelling petal and outside an attracting petal.

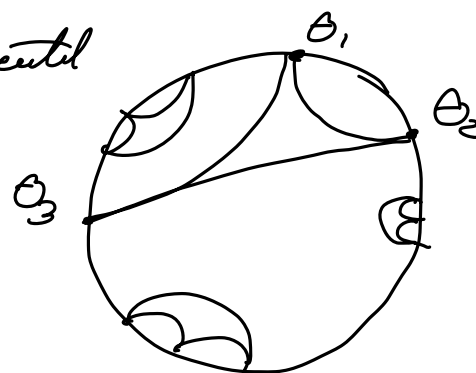


Coda. 1.)

Recall the homotopy $\mathbb{Z}$ associated $\mathbb{Z} K(f)$:

$$\theta_1 \sim \theta_2 \text{ if } \gamma(\theta_1) = \gamma(\theta_2)$$

$$\rightarrow E(\theta_1) \sim E(\theta_2),$$



$$\exists i, j \text{ s.t. } E\left(\frac{\theta_1 + i}{n}\right) \sim E\left(\frac{\theta_2 + j}{n}\right)$$

* $\mathcal{Z}$ is invariant under $E: \mathcal{S}' \rightarrow \mathcal{S}'$. To find $\mathcal{Z}$, take preimages.

2) If $\mathcal{O}$ is a repelling or parabolic periodic cycle, $\gamma^{-1}(\mathcal{O})$ is an orbit portrait:

• $\{A_1, \dots, A_k\}$, $E: A_i \xrightarrow{\sim} A_{i+1}$, preserve order, pairs A_i are pairwise disjoint.